Logic in Action

Chapter 9: Proofs

http://www.logicinaction.org/

Systems revised so far

Issues with the **tableau** method.

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- It is a *refutation* method.

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Issues with the **tableau** method.

- It is a *refutation* method.
- It does not follow the way humans reason.

Issues with the **presented derivation systems**.

- Proofs are not very natural (e.g., try to prove $\varphi \to \neg\neg\varphi$).
- They do not facilitate *conditional* reasoning.

The *deduction* property

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$$\Sigma, \varphi \models \psi \qquad \text{if and only if} \qquad \Sigma \models \varphi \rightarrow \psi$$

What if we can make assumptions?

Consider a proof for $\varphi \to \varphi$.

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- Using the derivation system presented in Chapter 2, the proof takes several steps.

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Consider a proof for $\varphi \rightarrow \varphi$.

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- But if we can make assumptions …

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1 φ

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- But if we can make assumptions …

1	φ	
2	φ	`repetition 1`

What if we can make assumptions?

Consider a proof for $\varphi \to \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions …

1	$\mid \varphi$	
2	$\mid \varphi$	`repetition 1`
3	$\varphi \to \varphi$	`deduction 1-2`

What if we can make assumptions?

Consider a proof for $\varphi \longrightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions …

1	φ	
2	φ	`repetition 1`
3	$\varphi \longrightarrow \varphi$	`deduction 1-2`

This is the main idea for **the deduction rule**.

The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

The deduction rule

Suppose you want to prove $\varphi \to \psi$.

- Assume φ.

φ

The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume φ.
- If after further steps

The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume φ.
- If after further steps
- you can prove ψ,

φ

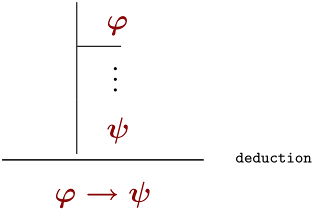
$\vdots$

ψ

The **deduction** rule

Suppose you want to prove $\varphi \to \psi$.

- Assume φ.
- If after further steps
- you can prove ψ,
- then you actually have $\varphi \to \psi$.



Recall

The three axioms for propositional logic

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1. $\varphi \rightarrow (\psi \rightarrow \varphi)$

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1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

Recall

The three axioms for propositional logic

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$
2. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
3. $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$

Proving the axioms (1)

The axiom

$$\varphi \to (\psi \to \varphi)$$

can be proved from *deduction*:

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$$\begin{array}{ll} 1 & \left| \underline{\ \varphi\ } \right. \end{array}$$

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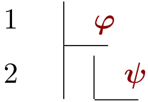

1 φ

2 ψ

Proving the axioms (1)

The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

1	φ	
2	ψ	
3	φ	`repetition 1`

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The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

1	φ	
2	$\quad \psi$	
3	$\quad \varphi$	`repetition 1`
4	$\psi \rightarrow \varphi$	`deduction 2-3`

Proving the axioms (1)

The axiom

$$\varphi \to (\psi \to \varphi)$$

can be proved from *deduction*:

1	$\mid\ \varphi$	
2	$\mid\ \mid\ \psi$	
3	$\mid\ \mid\ \varphi$	`repetition 1`
4	$\mid\ \psi \to \varphi$	`deduction 2-3`
5	$\varphi \to (\psi \to \varphi)$	`deduction 1-4`

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

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2 $\varphi \to \psi$

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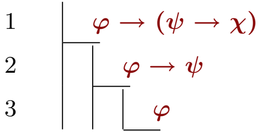

1 $\varphi \to (\psi \to \chi)$

2 $\varphi \to \psi$

3 φ

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\varphi \to \psi$	
3	φ	
4	ψ	`modus ponens 3,2`

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\varphi \to \psi$	
3	φ	
4	ψ	`modus ponens 3,2`
5	$\psi \to \chi$	`modus ponens 3,1`

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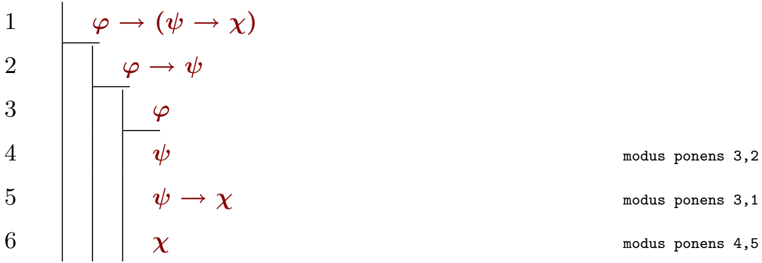

1	$\varphi \to (\psi \to \chi)$	
2	$\varphi \to \psi$	
3	φ	
4	ψ	`modus ponens 3,2`
5	$\psi \to \chi$	`modus ponens 3,1`
6	χ	`modus ponens 4,5`

Proving the axioms (2)

The axiom

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

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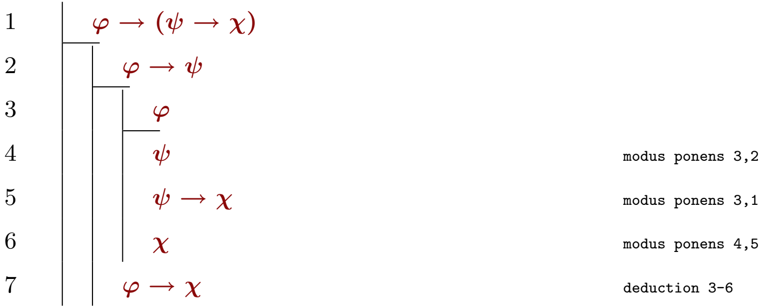

1	$\varphi \rightarrow (\psi \rightarrow \chi)$	
2	$\varphi \rightarrow \psi$	
3	φ	
4	ψ	`modus ponens 3,2`
5	$\psi \rightarrow \chi$	`modus ponens 3,1`
6	χ	`modus ponens 4,5`
7	$\varphi \rightarrow \chi$	`deduction 3-6`

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2	$\varphi \to \psi$	
3	φ	
4	ψ	`modus ponens 3,2`
5	$\psi \to \chi$	`modus ponens 3,1`
6	χ	`modus ponens 4,5`
7	$\varphi \to \chi$	`deduction 3-6`
8	$(\varphi \to \psi) \to (\varphi \to \chi)$	`deduction 2-7`

Proving the axioms (2)

The axiom

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:

1	$\varphi \to (\psi \to \chi)$	
2	$\varphi \to \psi$	
3	φ	
4	ψ	`modus ponens 3,2`
5	$\psi \to \chi$	`modus ponens 3,1`
6	χ	`modus ponens 4,5`
7	$\varphi \to \chi$	`deduction 3-6`
8	$(\varphi \to \psi) \to (\varphi \to \chi)$	`deduction 2-7`
9	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	`deduction 1-8`

We need more

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

cannot be proved from *modus ponens* and *deduction*.

We need more

The axiom

$$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$$

cannot be proved from *modus ponens* and *deduction*.

We need a way to deal with negations.

The **refutation** rule

Suppose you want to prove φ.

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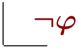
Suppose you want to prove φ.

- Assume $\neg\varphi$.

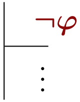


$$\neg\varphi$$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps



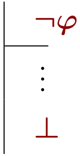
$\neg\varphi$

$\vdots$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction $\bot$,



$\neg\varphi$

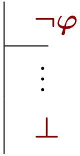
$\vdots$

$\bot$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction $\bot$,
- then $\neg\varphi$ cannot be true



$\neg\varphi$

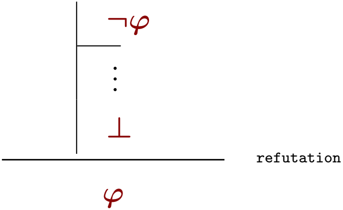
$\vdots$

$\bot$

The **refutation** rule

Suppose you want to prove φ.

- Assume $\neg\varphi$.
- If after further steps
- you can prove a contradiction $\bot$,
- then $\neg\varphi$ cannot be true
- so you actually have φ.



$\neg\varphi$

$\vdots$

$\bot$

`refutation`

φ

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

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2 $\quad\quad \psi$

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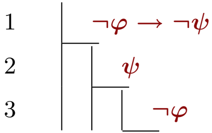

1 $\neg\varphi \to \neg\psi$

2 ψ

3 $\neg\varphi$

Proving the axioms (3)

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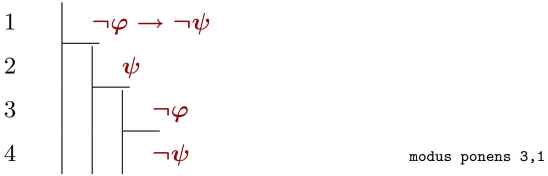


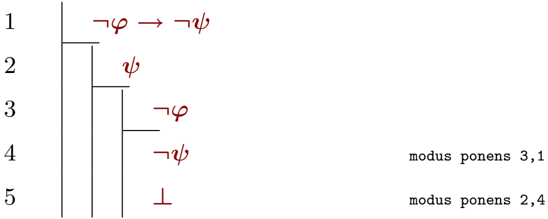
1	$\neg\varphi \to \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	`modus ponens 3,1`

Proving the axioms (3)

The axiom

$$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:



1	$\neg\varphi \rightarrow \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	`modus ponens 3,1`
5	$\bot$	`modus ponens 2,4`

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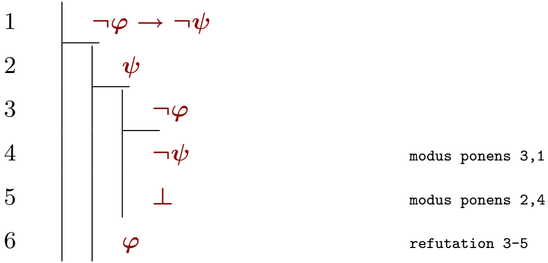

1	$\neg\varphi \rightarrow \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	`modus ponens 3,1`
5	$\bot$	`modus ponens 2,4`
6	φ	`refutation 3-5`

Proving the axioms (3)

The axiom

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1	$\mid \neg\varphi \to \neg\psi$	
2	$\mid \; \mid \psi$	
3	$\mid \; \mid \; \mid \neg\varphi$	
4	$\mid \; \mid \; \mid \neg\psi$	`modus ponens 3,1`
5	$\mid \; \mid \; \mid \bot$	`modus ponens 2,4`
6	$\mid \; \mid \varphi$	`refutation 3-5`
7	$\mid \psi \to \varphi$	`deduction 2-6`

Proving the axioms (3)

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1	$\neg\varphi \to \neg\psi$	
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7	$\psi \to \varphi$	`deduction 2-6`
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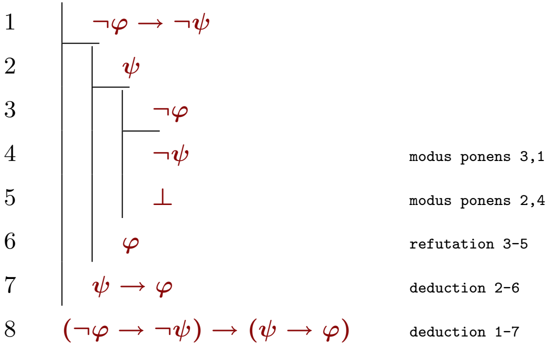

1	$\neg\varphi \rightarrow \neg\psi$	
2	ψ	
3	$\neg\varphi$	
4	$\neg\psi$	`modus ponens 3,1`
5	$\bot$	`modus ponens 2,4`
6	φ	`refutation 3-5`
7	$\psi \rightarrow \varphi$	`deduction 2-6`
8	$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$	`deduction 1-7`

For step 5, note that $\neg\psi$ can be seen as an abbreviation of $\psi \rightarrow \bot$.

So . . .

$$\frac{\varphi\ ,\ \varphi \rightarrow \psi}{\psi}\quad \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi}\quad \texttt{deduction} \qquad\qquad \frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\quad \texttt{refutation}$$

So . . .

$$\frac{\varphi\ ,\ \varphi \to \psi}{\psi}\ \texttt{modus ponens}$$

$$\frac{\begin{array}{|l}\varphi\\ \hline \vdots\\ \psi\end{array}}{\varphi \to \psi}\ \texttt{deduction} \qquad \frac{\begin{array}{|l}\neg\varphi\\ \hline \vdots\\ \bot\end{array}}{\varphi}\ \texttt{refutation}$$

The *modus ponens*, *deduction* and *refutation* rules are a complete system for propositional logic.

To facilitate things . . .

To facilitate things . . .

- Natural deduction introduces rules to manipulate all the connectives in an easy way.

For implication →

For implication $\rightarrow$

$$\frac{\varphi \,,\, \varphi \rightarrow \psi}{\psi} \quad \texttt{modus ponens}$$

For implication →

$$\frac{\varphi\ ,\ \varphi \to \psi}{\psi}$$ `modus ponens`

$$\mathbf{E}_{\to}$$

For implication →

$$\frac{\varphi \, , \, \varphi \to \psi}{\psi} \quad \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \to \psi} \quad \texttt{deduction}$$

$$\mathbf{E}_{\to}$$

For implication →

$$\frac{\varphi\ ,\ \varphi \to \psi}{\psi}\ \texttt{modus ponens}$$

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \to \psi}\ \texttt{deduction}$$

$\mathbf{E}_{\to}$ $\mathbf{I}_{\to}$

For negation ¬

For negation ¬

$$\frac{\neg\varphi \,,\, \varphi}{\bot}$$

For negation ¬

$$\frac{\neg\varphi \, , \, \varphi}{\bot}$$

$\mathbf{E}_{\neg}$

For negation ¬

$$\frac{\neg\varphi \, , \, \varphi}{\bot}$$

$$\frac{\begin{array}{|l}\neg\varphi \\ \hline \vdots \\ \bot\end{array}}{\varphi}\ \texttt{refutation}$$

$\mathbf{E}_{\neg}$

For negation ¬

$$\frac{\neg\varphi \,,\, \varphi}{\bot}$$

$$\mathbf{E}_{\neg}$$

$$\frac{\begin{array}{|l} \neg\varphi \\ \hline \vdots \\ \bot \end{array}}{\varphi}\quad \texttt{refutation}$$

$$\mathbf{I}_{\neg}$$

For conjunction $\wedge$

For conjunction $\wedge$

$$\frac{\varphi \wedge \psi}{\varphi}$$

$$\frac{\varphi \wedge \psi}{\psi}$$

For conjunction $\wedge$

$$\frac{\varphi \wedge \psi}{\varphi}$$

$$\frac{\varphi \wedge \psi}{\psi}$$

$\mathbf{E}_{\wedge}$

For conjunction $\wedge$

$$\frac{\varphi \wedge \psi}{\varphi}$$

$$\frac{\varphi \wedge \psi}{\psi}$$

$$\frac{\varphi \, , \, \psi}{\varphi \wedge \psi}$$

$\mathbf{E}_{\wedge}$

For conjunction $\wedge$

$\dfrac{\varphi \wedge \psi}{\varphi}$ $\dfrac{\varphi \wedge \psi}{\psi}$	$\dfrac{\varphi \, , \, \psi}{\varphi \wedge \psi}$
$\mathbf{E}_{\wedge}$	$\mathbf{I}_{\wedge}$

For disjunction $\vee$

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$$\dfrac{\varphi \vee \psi\,, \qquad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \qquad , \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

For disjunction $\vee$

$$\frac{\varphi \vee \psi\,, \qquad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \qquad , \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

$\mathbf{E}_{\vee}$

For disjunction $\vee$

$$\frac{\varphi \vee \psi\,, \qquad \begin{array}{|l}\varphi\\ \hline \vdots\\ \chi\end{array} \qquad , \qquad \begin{array}{|l}\psi\\ \hline \vdots\\ \chi\end{array}}{\chi}$$

$$\frac{\varphi}{\varphi \vee \psi}$$

$$\frac{\psi}{\varphi \vee \psi}$$

$\mathbf{E}_{\vee}$

For disjunction $\vee$

$$\frac{\varphi \vee \psi\,, \qquad \begin{array}{|l} \varphi \\ \hline \vdots \\ \chi \end{array} \qquad , \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \chi \end{array}}{\chi}$$

$$\frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi}$$

$\mathbf{E}_\vee$ $\qquad$ $\mathbf{I}_\vee$

For *predicate* logic

In order to present *introduction* and *elimination* rules for both ∀ and ∃, we need to recall two notions.

For *predicate* logic

In order to present *introduction* and *elimination* rules for both $\forall$ and $\exists$, we need to recall two notions.

- **Bounded variable**.

For *predicate* logic

In order to present *introduction* and *elimination* rules for both $\forall$ and $\exists$, we need to recall two notions.

- **Bounded variable**.
- **Substitution of a variable for a term in a formula**.

Bounded variable

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- **Scope of a quantifier.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the subformula φ is said to be **the scope** of the quantifier $\forall$ ($\exists$).

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- **Binding a variable.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the quantifier $\forall$ ($\exists$) **binds** any occurrence of x in φ that is not bounded by another quantifier inside φ.

Bounded variable

- **Scope of a quantifier.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the subformula φ is said to be **the scope** of the quantifier $\forall$ ($\exists$).

- **Binding a variable.** In a formula of the form $\forall x\varphi$ ($\exists x\varphi$), the quantifier $\forall$ ($\exists$) **binds** any occurrence of x in φ that is not bounded by another quantifier inside φ.

- **Bounded variable**. An occurrence of a variable x is **bounded** in a formula φ if there is a quantifier in φ that binds it.

Substitution (1)

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- **Substitution inside a term.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

$$(s)^y_t$$

Substitution (1)

- **Substitution inside a term.** Replacing the occurrences of the variable y for the term t inside the **term** s produces the **term** denoted by

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- Formally,

For a **constant**: $(c)^{y}_{t} := c$

For a **variable**: $\begin{cases} (x)^{y}_{t} := x & \text{for } x \text{ different from } y \\ (y)^{y}_{t} := t & \end{cases}$

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Examples:

$$(a)^x_c := a$$
$$(x)^y_a := x$$
$$(z)^z_y := y$$

Substitution (2)

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- **Substitution inside a <u>formula</u>.** Replacing the **free** occurrences of the variable y for the term t inside the **formula** φ produces the **formula** denoted by

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$$\begin{aligned}(Pt_1 \cdots t_n)^y_t &:= P(t_1)^y_t \cdots (t_n)^y_t\\ (\neg\varphi)^y_t &:= \neg(\varphi)^y_t\\ (\varphi \wedge \psi)^y_t &:= (\varphi)^y_t \wedge (\psi)^y_t\\ (\varphi \vee \psi)^y_t &:= (\varphi)^y_t \vee (\psi)^y_t\\ (\varphi \rightarrow \psi)^y_t &:= (\varphi)^y_t \rightarrow (\psi)^y_t\\ (\varphi \leftrightarrow \psi)^y_t &:= (\varphi)^y_t \leftrightarrow (\psi)^y_t\end{aligned}$$

$$\begin{cases}(\forall x\varphi)^y_t := \forall x(\varphi)^y_t\\ (\forall y\varphi)^y_t := \forall y\varphi\end{cases}$$

$$\begin{cases}(\exists x\varphi)^y_t := \exists x(\varphi)^y_t\\ (\exists y\varphi)^y_t := \exists y\varphi\end{cases}$$

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$$\frac{\begin{array}{l|l} u & \\ \hline & \vdots \\ & (\varphi)^x_u \end{array}}{\forall x\,\varphi}$$

for u a special symbol not used anywhere else in the proof

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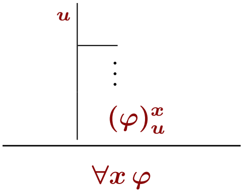

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For the existential quantifier ∃

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$$
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$$

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for any term t.

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